Optimum phase measurement in the presence of noise

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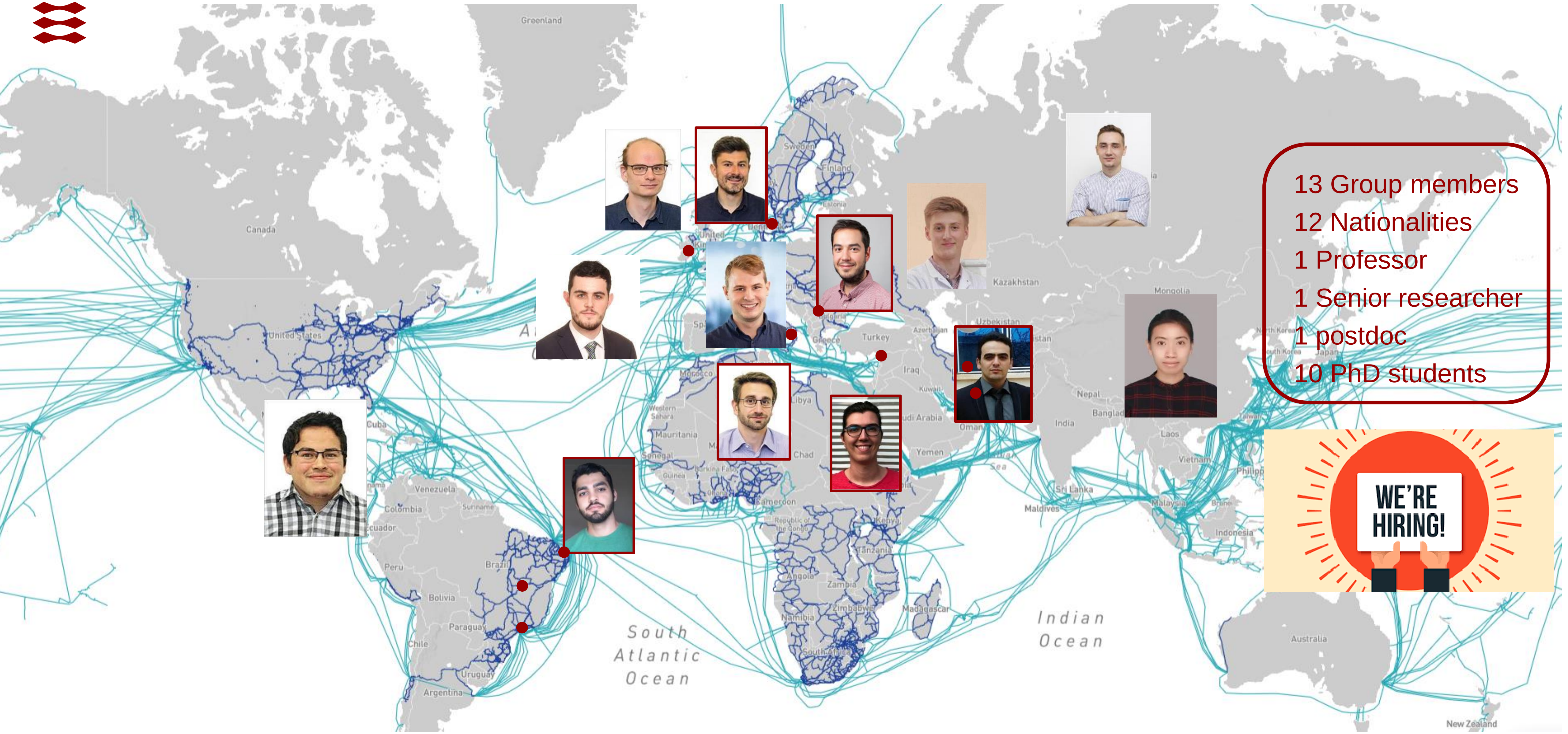
Machine Learning in Photonic Systems Group



13 Group members
12 Nationalities
1 Professor
1 Senior researcher
1 postdoc
10 PhD students

Research topics and collaborations

Machine learning enabled ultra-wideband amplifier design

Unifying framework for laser and frequency comb noise characterization

Machine learning techniques for communicate over complex channels

Optical technologies to accelerate AI

Machine learning enabled optical fibre systems

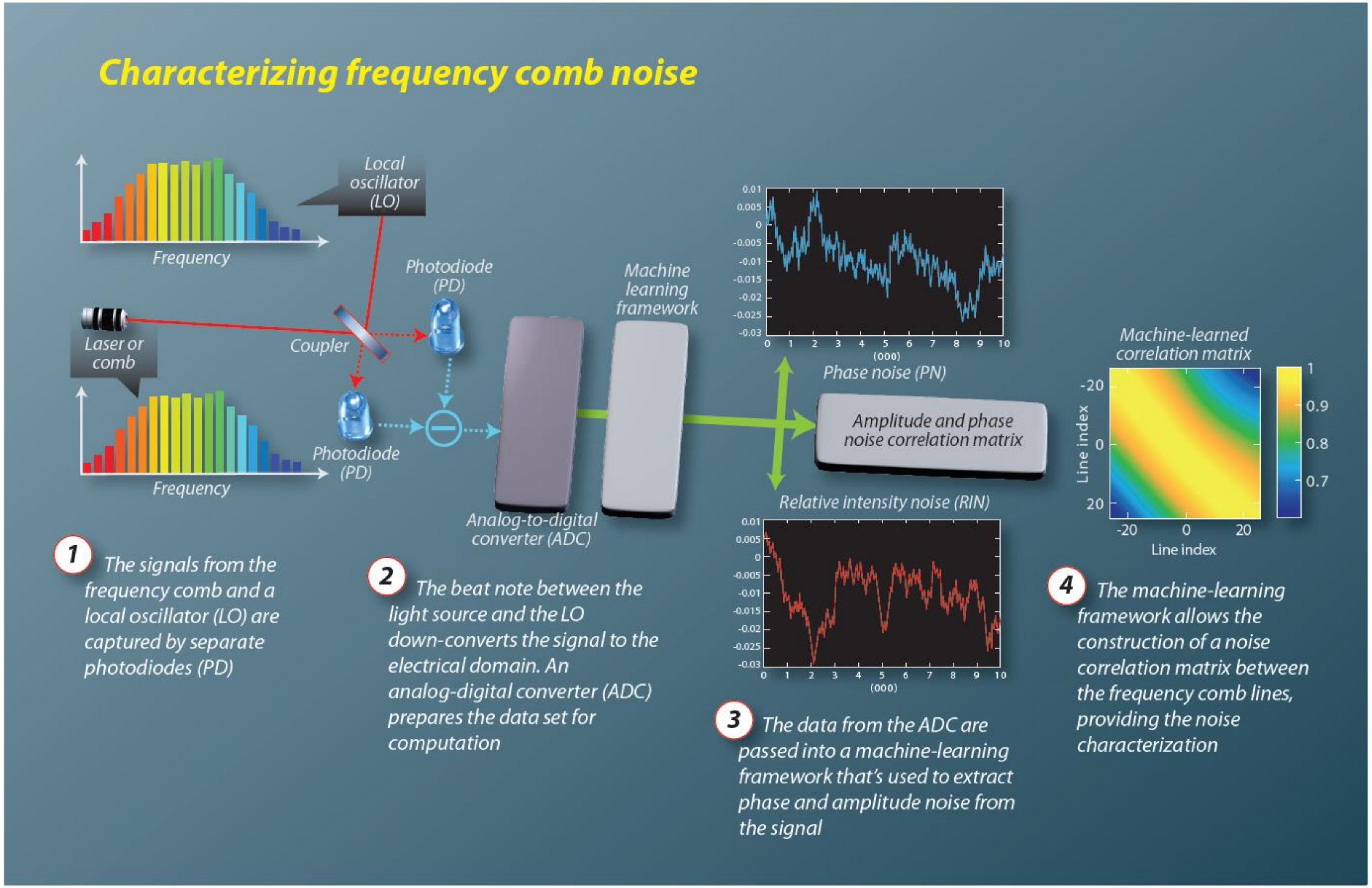


Illustration by Phil Saunders

[1] D. Zibar et al., PTL 2019

[2] G. Brajatto et al, Optics Express, 2020, [3] D Zibar et al Optica 2021, [4] D. Zibar Optics & Photonics News 2020

Research question to be answered

What is the impact of amplifier noise on signal phase?

- Relevant for building high-power ultra-narrow linewidth lasers
- Relevant for generation of optical frequency combs
- Relevant for transmission of frequency standards
- Relevant for noise characterization of lasers and frequency combs

Surprisingly few works on the topic and no common agreement

SPECTRAL BROADENING DUE TO FIBRE AMPLIFIER PHASE NOISE

ELECTRONICS LETTERS 29th March 1990 Vol. 26 No. 7

G. J. COWLE†
P. R. MORKEL
R. I. LAMING
D. N. PAYNE

1554 IEEE JOURNAL OF QUANTUM ELECTRONICS, VOL. 34, NO. 9, SEPTEMBER 1998

Novel Aspects of Spectral Broadening Due to Fiber Amplifier Phase Noise

Lothar Möller

IEEE JOURNAL ON SELECTED TOPICS IN QUANTUM ELECTRONICS, VOL. 7, NO. 1, JANUARY/FEBRUARY 2001 49

New Investigations on the Effect of Fiber Amplifier Phase Noise

Etienne Rochat and René Dändliker

2610 Vol. 59, No. 8 / 10 March 2020 / *Applied Optics* **Research Article**

applied optics

Influence of amplified spontaneous emission on laser linewidth in a fiber amplifier

Mingyuan Xue,[1,2] Cunxiao Gao,[1,*] Linquan Niu,[1] Shaolan Zhu,[1] and Chuandong Sun[1]

DTU

1604 *PROCEEDINGS OF THE IRE* July

The Fundamental Noise Limit of Linear Amplifiers*

H. HEFFNER†, FELLOW, IRE

* Received January 8, 1962; revised manuscript received, April 30, 1962.

PHYSICAL REVIEW VOLUME 128, NUMBER 5 DECEMBER 1, 1962

Quantum Noise in Linear Amplifiers

H. A. HAUS
Electrical Engineering Department and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts

AND

J. A. MULLEN
Research Division, Raytheon Company, Waltham, Massachusetts
(Received May 10, 1962; revised manuscript received August 23, 1962)

Both papers derive that *power* and *phase fluctuation* due to amplifier noise are given by:

$$\boxed{\Delta P = (G-1)h\nu B}$$

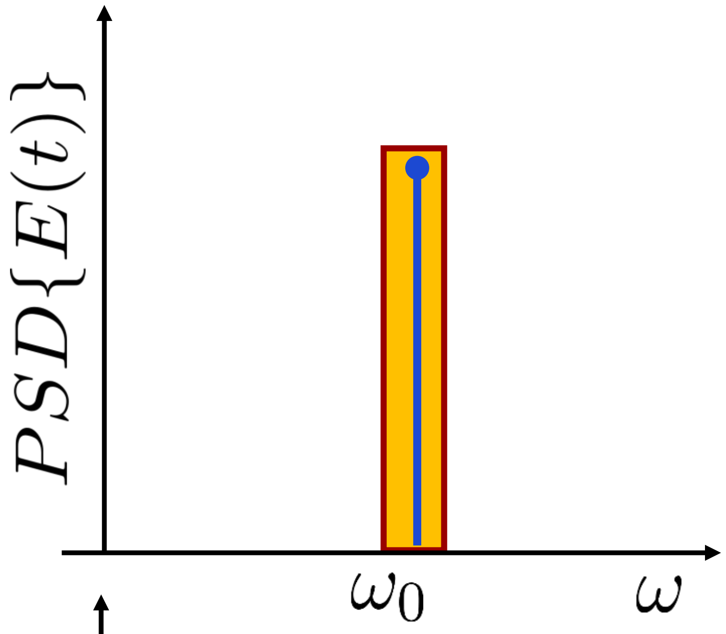
$$\boxed{\Delta\phi = \frac{(G-1)h\nu B}{2P}}$$

Heffner and Haus assume input to the amplifier contains single frequency
(Lasers have amplitude and phase noise)

Assumptions by Heffner and Haus

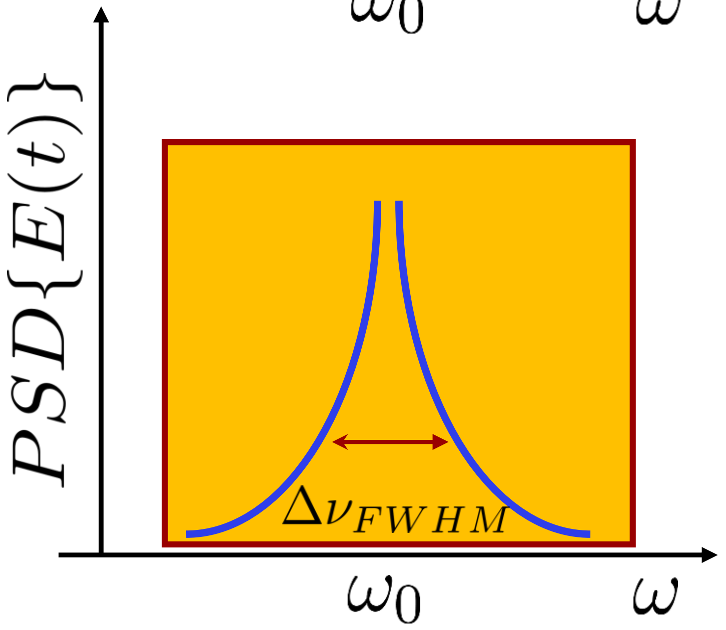
Constant phase signal:

$$E(t) = \sqrt{P_0}\sin[\omega_0 t + \phi_0]$$



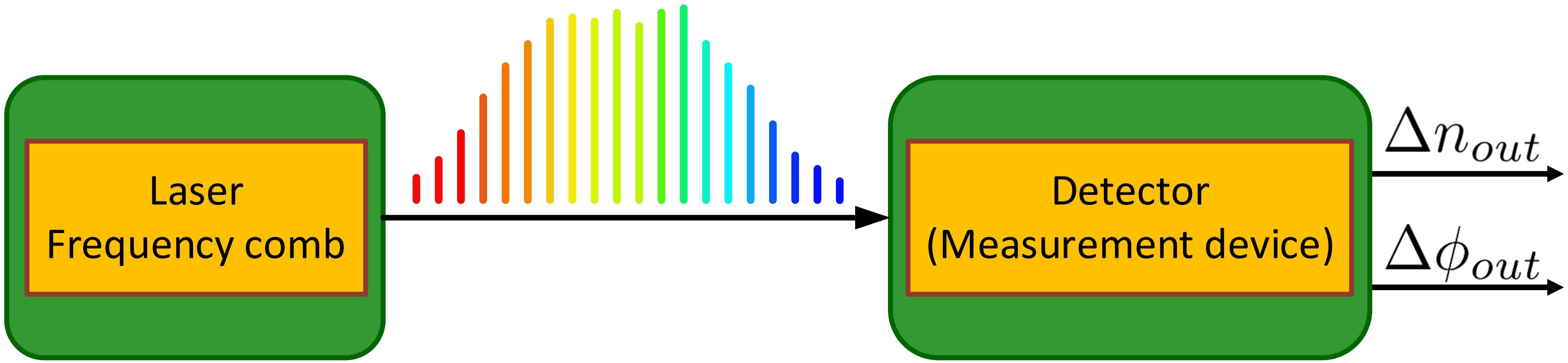
In practice amplitude and phase are (randomly) time-varying :

$$E(t) = \sqrt{P_0(1+\alpha(t))}\sin[\omega_0 t + \phi_0(t)]$$



The implication of time-varying amplitude and phase:
Measurement bandwidth needs to be carefully chosen

Ultimate laser stability conditioned by measurement precision



Heisenberg uncertainty sets limit on how accurately *photon number* and *phase* can be measured:

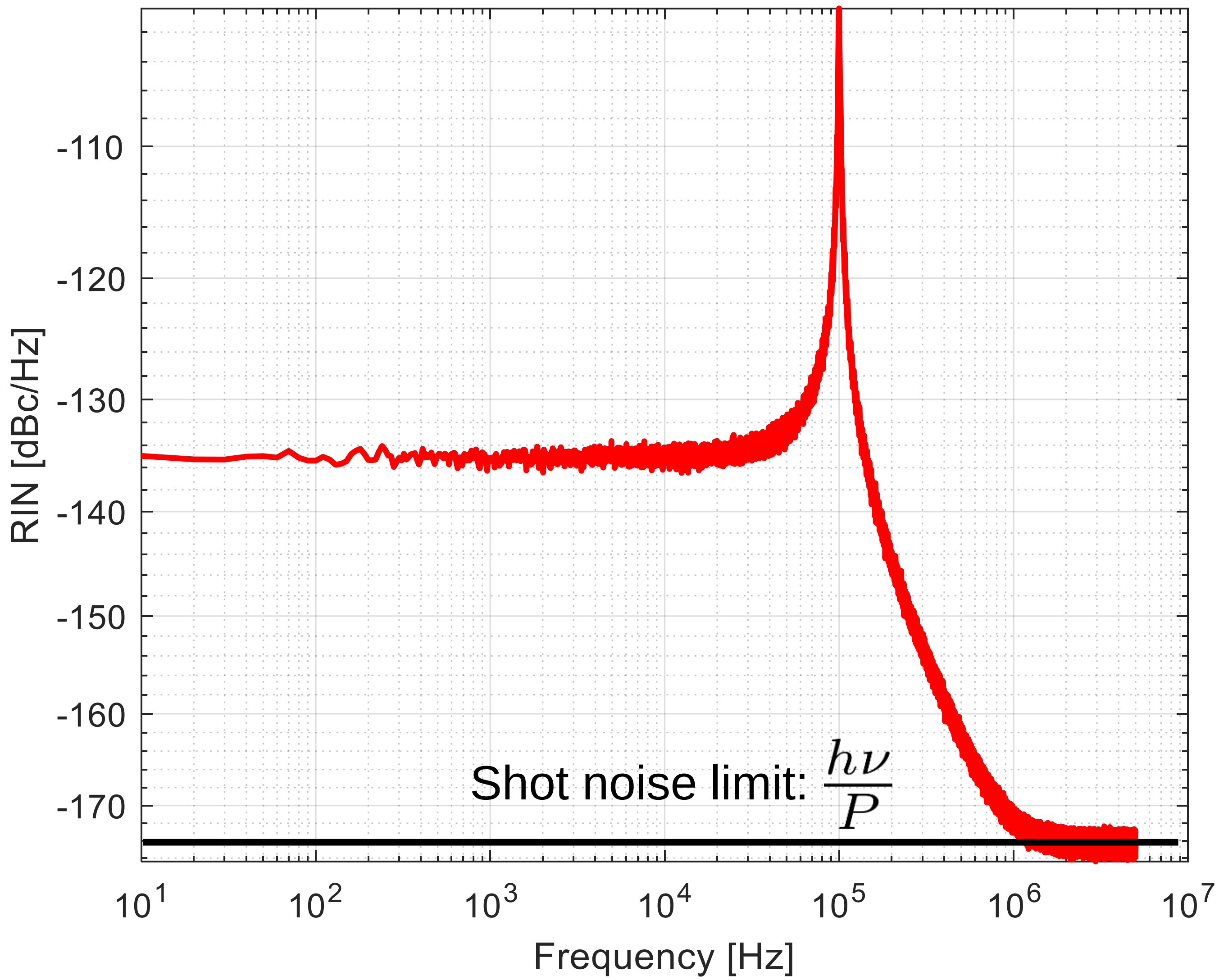
$$\boxed{\Delta n_{out} \Delta \phi_{out} \geq \frac{1}{2}}$$

In spectral domain *Heisenberg uncertainty* translates to:

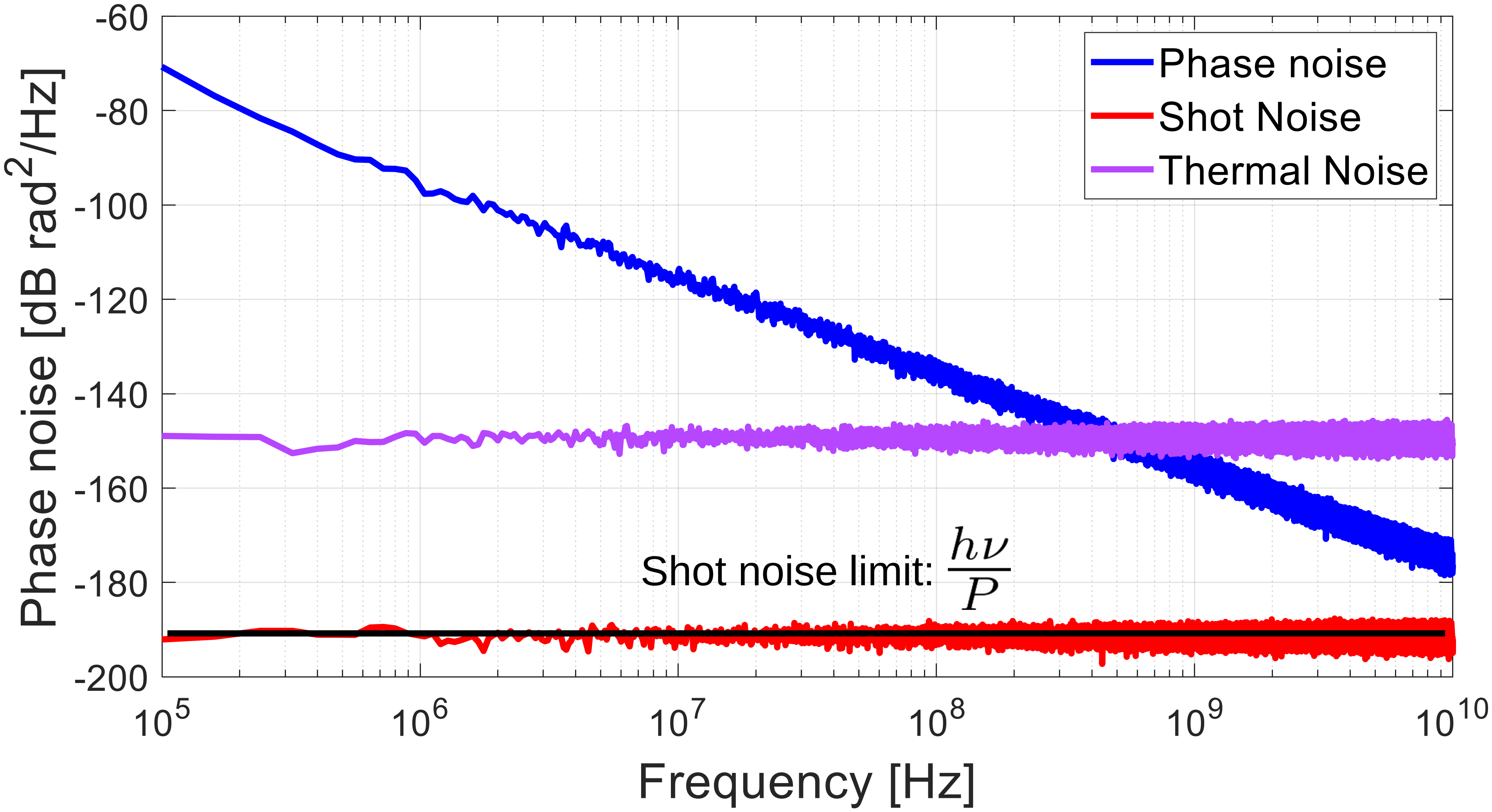
$$\boxed{S_{RIN}(f) S_{\phi}(f) \geq \left(\frac{h\nu}{2P}\right)^2}$$

A. Yariv, SPIE 1995

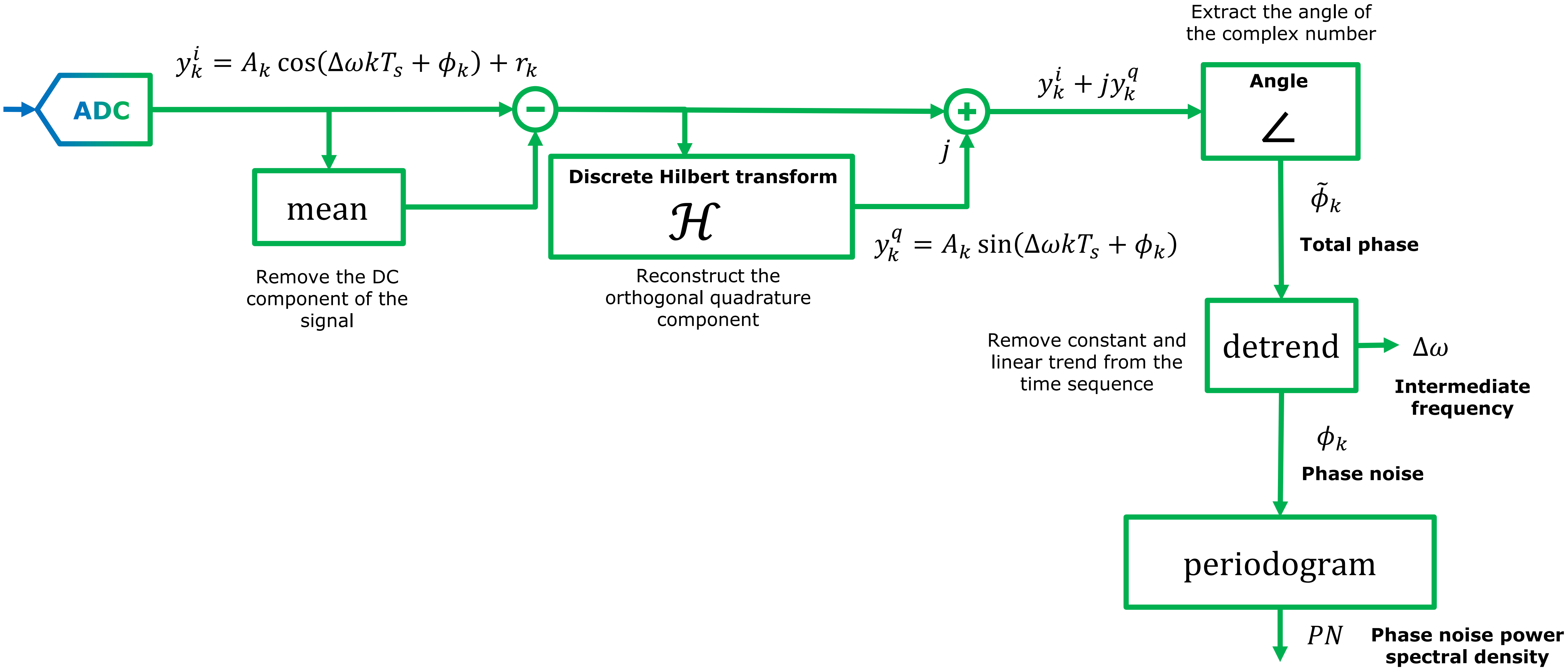
Limit on measuring Relative Intensity Noise (RIN)



Limit on measuring phase noise



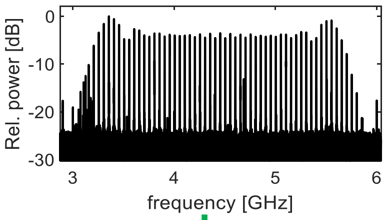
Conventional phase measurement

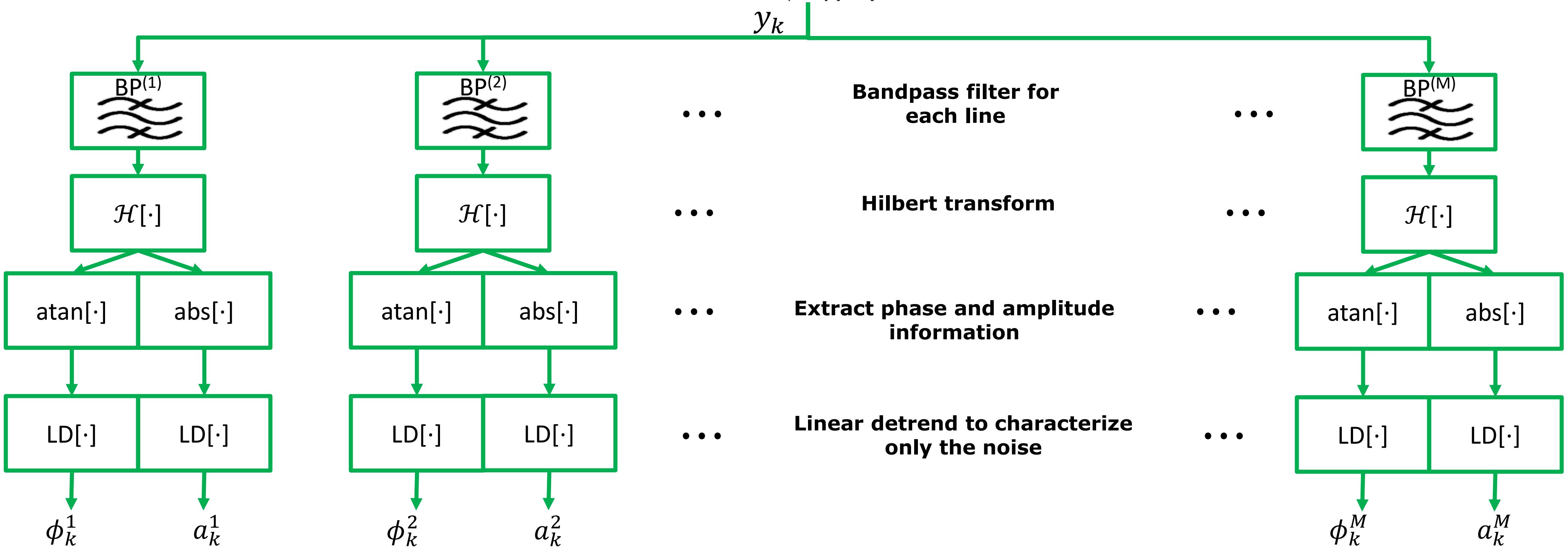


Conventional phase noise extraction

From the downconverted comb, extraction of amplitude and phase noise

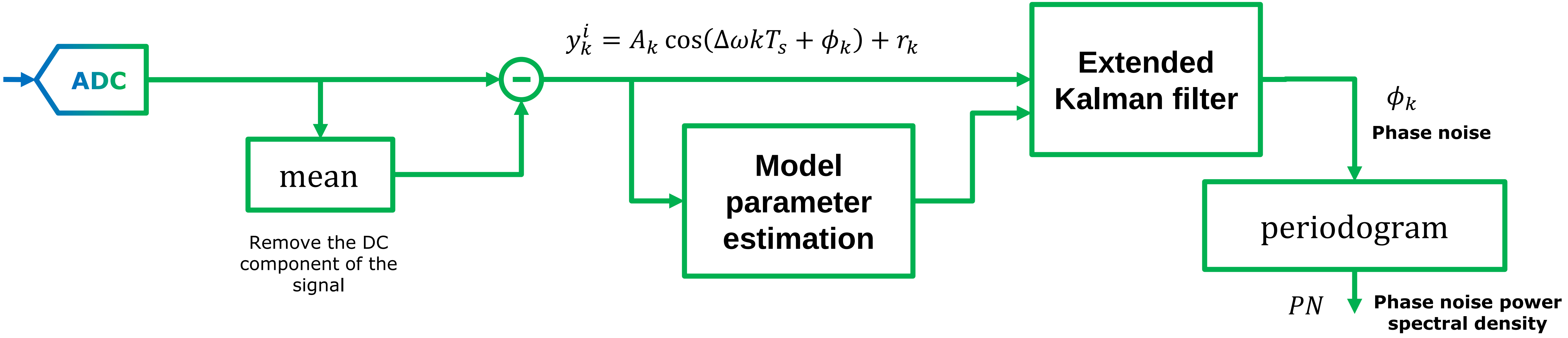
Parallel processing of all the frequency lines (generalization of a single line)





Problem! Measurement noise affect the comb noise estimation

Bayesian filtering based phase extraction



The model we are using

$$\phi_k = \phi_{k-1} + q_k^{\phi} \qquad q_k^{\phi} \sim \mathcal{N}\left(0, \sigma_{\phi}^2\right)$$

$$y_k^i = A\cos(\Delta\omega k T_s + \phi_k) + r_k \qquad r_k \sim \mathcal{N}(0, \sigma_r^2)$$

The parameters:

σ_{ϕ}^2 **Phase noise variance**

σ_r^2 **Measurement noise variance**

$\Delta\omega$ **Intermediate angular frequency**

A **Average signal amplitude**

Bayesian filtering for joint amplitude and phase noise estimation

Hidden state: phase and amplitude noise of all lines

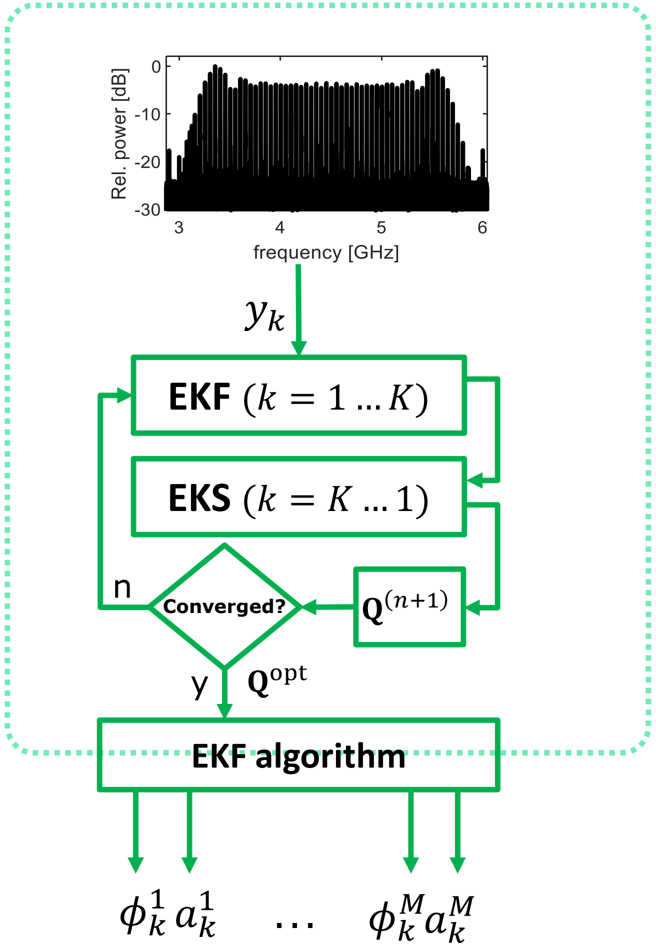
Phase and amplitude model: **Multidimensional Gaussian random walk**

$$\boldsymbol{\phi}_k = \begin{bmatrix} \phi_k^1 \\ \phi_k^2 \\ \vdots \\ \phi_k^M \end{bmatrix} \qquad \delta\boldsymbol{A}_k = \begin{bmatrix} \delta A_k^1 \\ \delta A_k^2 \\ \vdots \\ \delta A_k^M \end{bmatrix}$$

With M lines, we have M phase noise sequences and M amplitude noise sequences

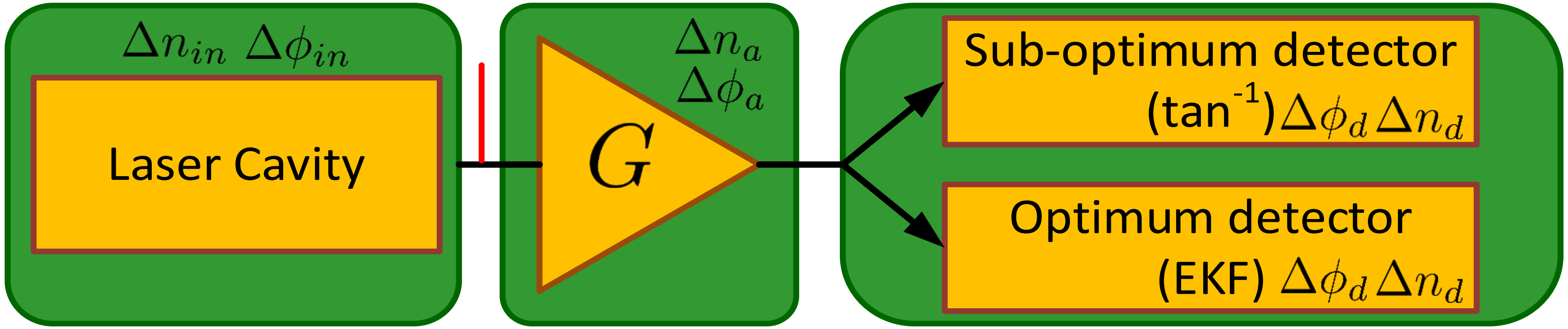
$$\begin{bmatrix} \boldsymbol{\phi}_k \\ \delta\boldsymbol{A}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{\phi}_{k-1} \\ \delta\boldsymbol{A}_{k-1} \end{bmatrix} + \boldsymbol{q}_{k-1}, \qquad \text{with } \boldsymbol{q}_{k-1} \sim \mathcal{N}(0, \mathbf{Q}) \text{ ,} \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_\phi & \mathbf{Q}_{\phi A} \\ \mathbf{Q}_{A\phi} & \mathbf{Q}_A \end{bmatrix}$$

$$y_k = \sum_{m=1}^{M} \bar{A}^m (1 + \delta A_k^m) \cos(\Delta\omega_m k T_S + \phi_k^m) + n_k$$



[1] G. Brajato et al, Optics Express 2020

The concept of optimum detector



$$\Delta n_{out}^2 = \Delta n_a^2 + \Delta n_d^2$$
$$\Delta\phi_{out}^2 = \Delta\phi_a^2 + \Delta\phi_d^2$$

Ultimate performance limit governed by:

$$\boxed{\Delta n_{out}\Delta\phi_{out} \geq \frac{1}{2}}$$

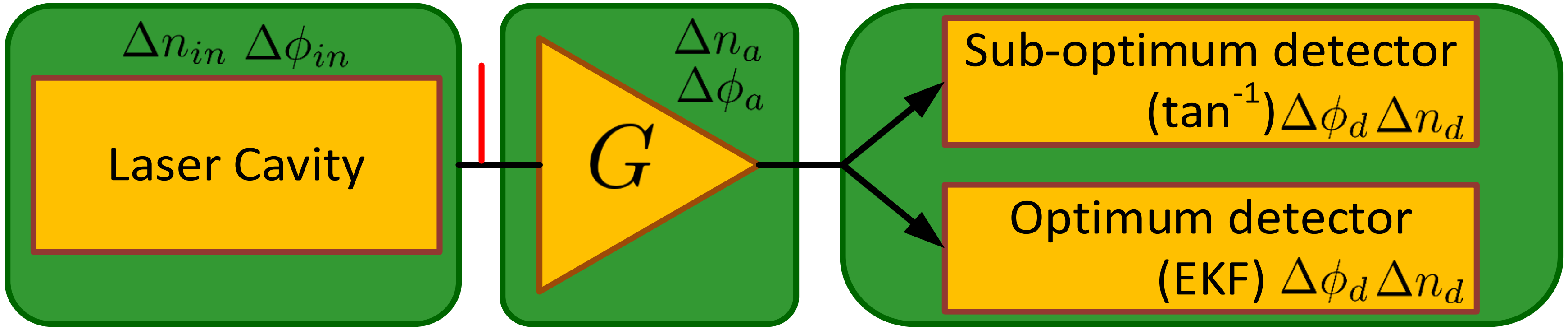
Heisenberg uncertainty limit reached when:

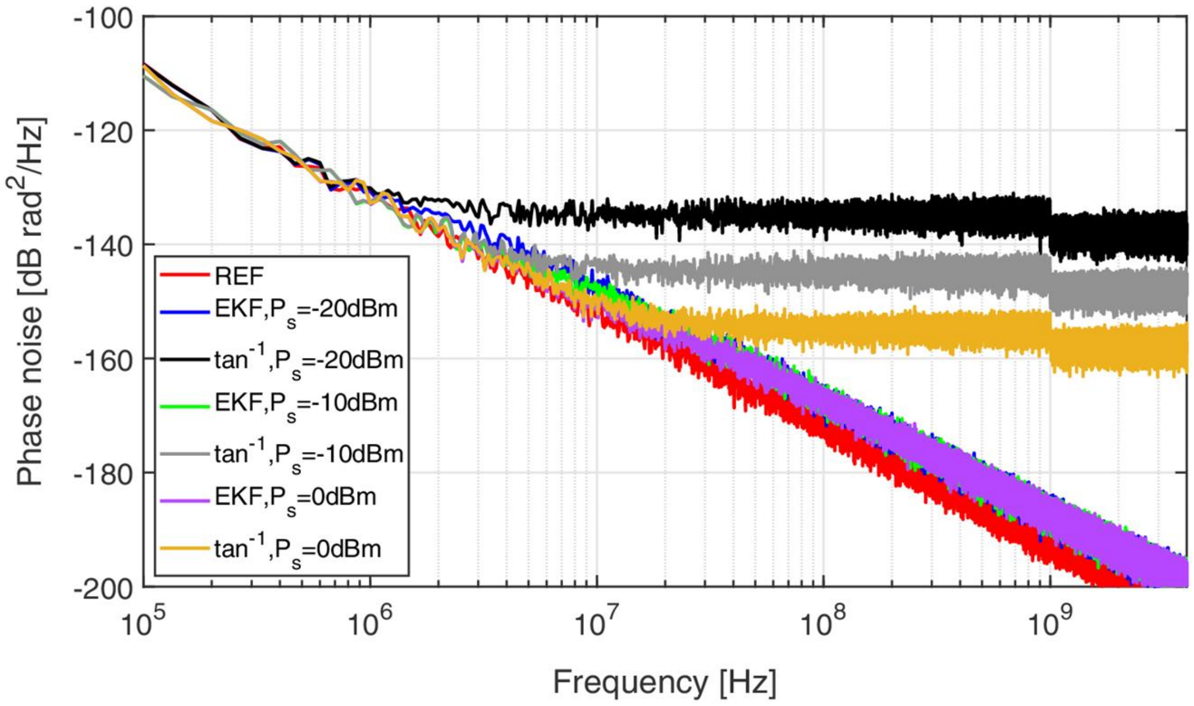
$$\boxed{\frac{\Delta n_d}{\Delta\phi_d} = \frac{\Delta n_a}{\Delta\phi_a}}$$

Detector uncertainty needs to be *matched* to amplifier uncertainty (optimum detector)

H. Heffner, 1962

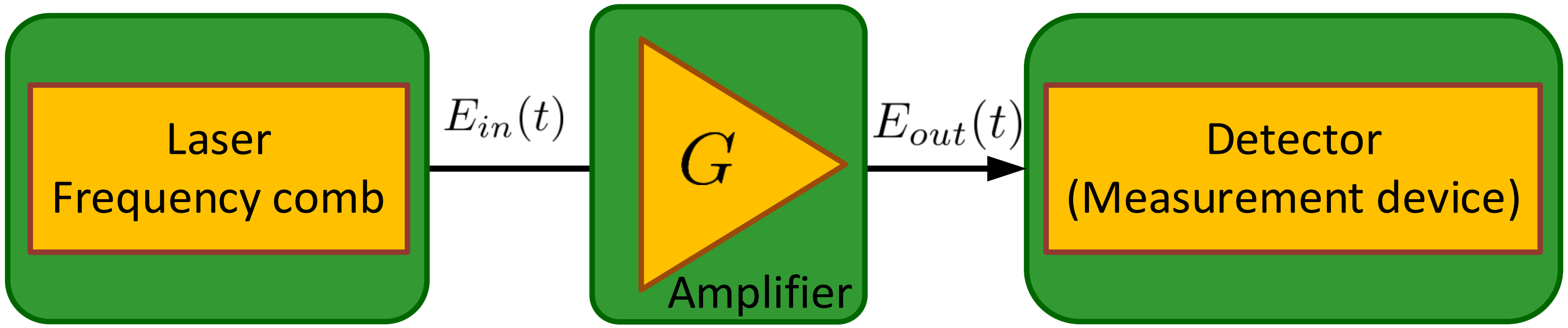
The importance of optimum detector





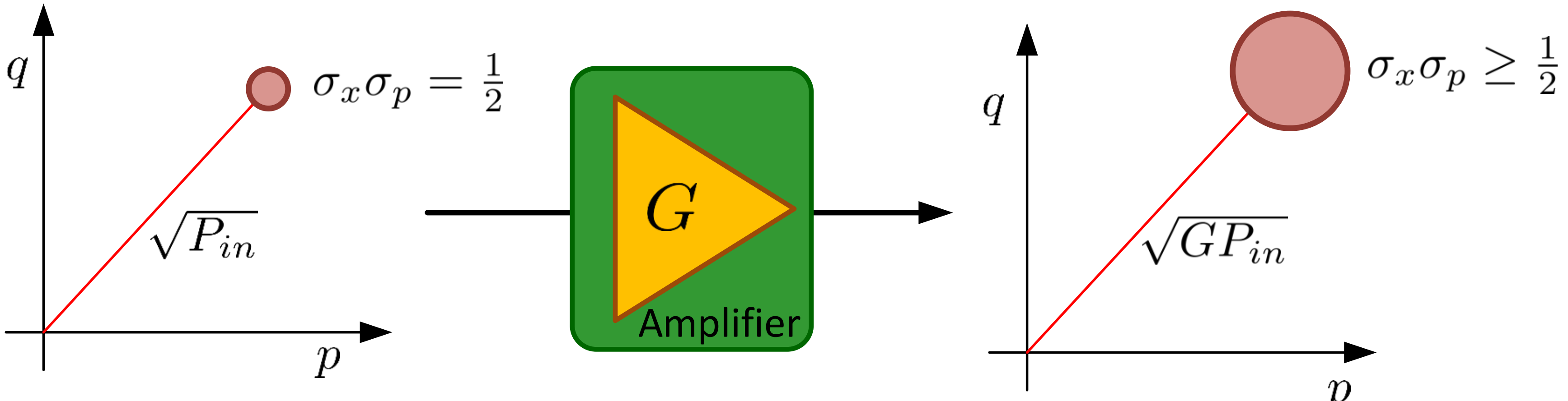
Darko Zibar, Jens E. Pedersen, Poul Varming, Giovanni Brajato, Francesco Da Ros, "Approaching optimum phase measurement in the presence of amplifier noise," Optica **8,** 1262-1267 (2021);

Minimum noise added by the amplifier



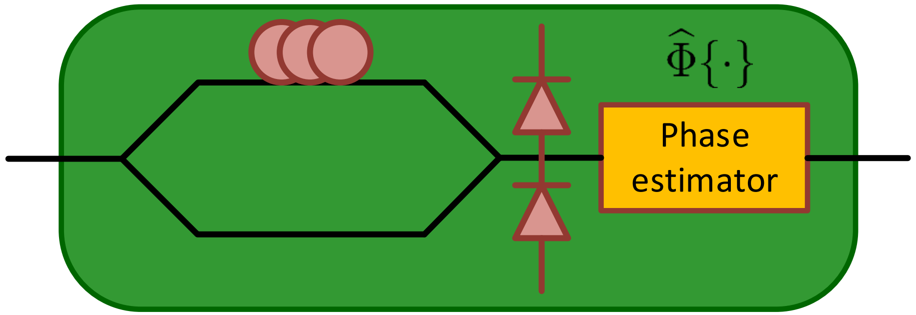
$$E_{in}(t) = \sqrt{P_{in}}\,\sin[\omega t + \phi_0] \qquad E_{out}(t) = \sqrt{GP_{in}}\,\sin[\omega t + \phi_0] + N(t)$$

$N(t)$: Gaussian noise with zero mean and standard deviation $\Delta P = (G-1)h\nu B$

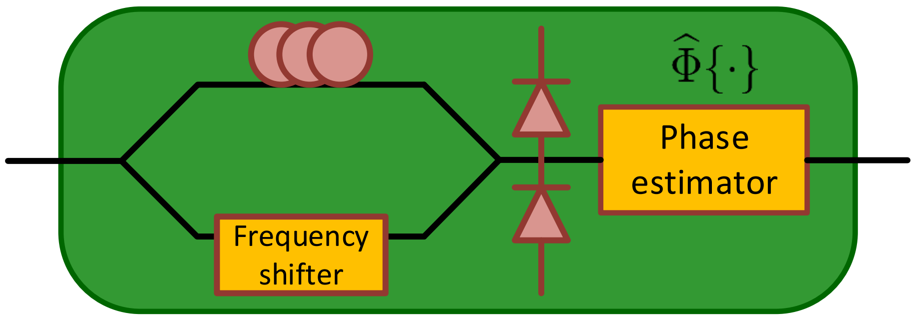


E. Desurvire, 1994

Quantum limited phase detection (the best we can do)

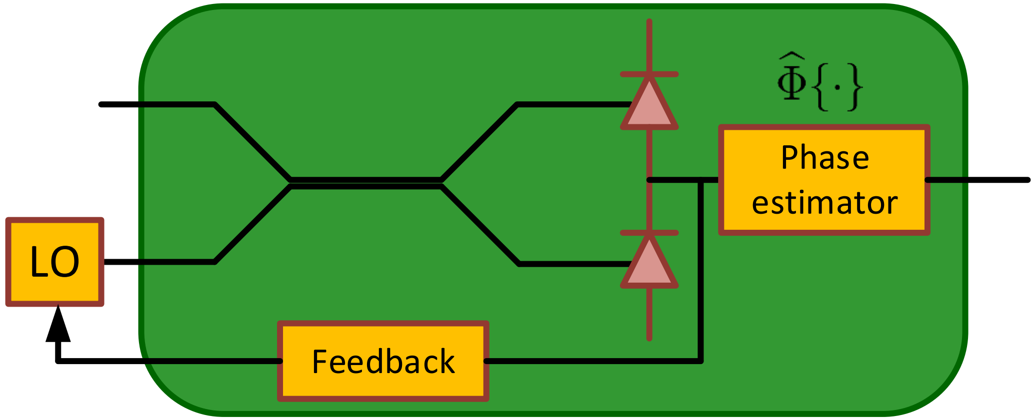


$$\Delta\phi_{ql}^2 = \frac{1}{\sqrt{N_p}}$$

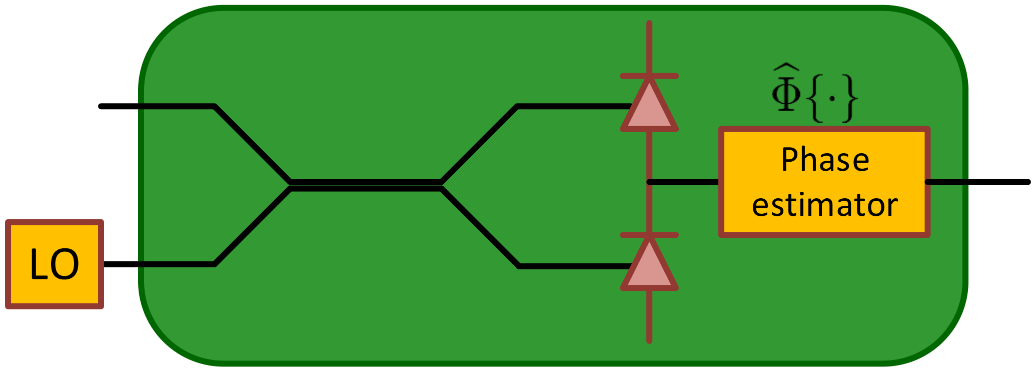


$$\Delta\phi_{ql}^2 = \frac{1}{\sqrt{N_p/2}}$$

$$N_p = \frac{P_{in}}{h\nu\pi\Delta\nu_{FWHM}}$$



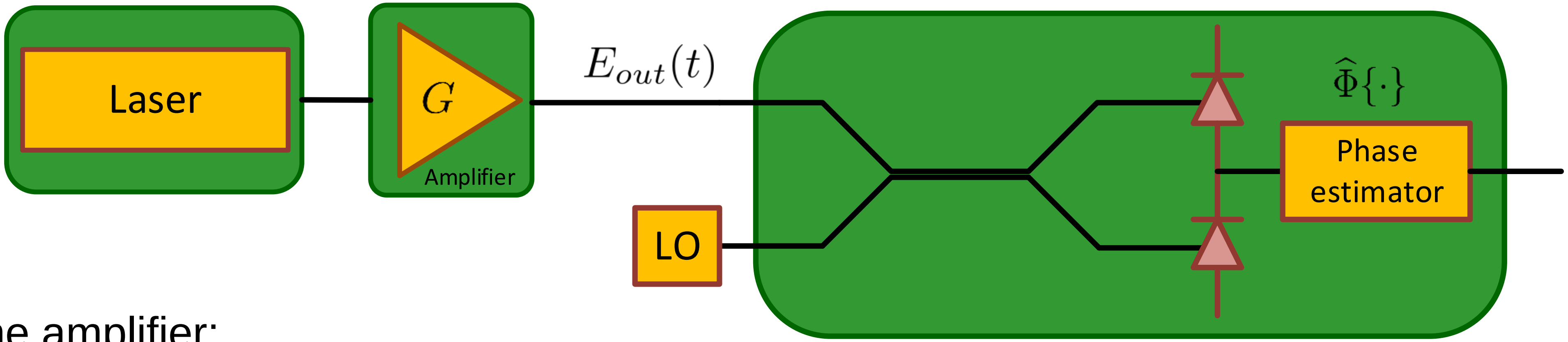
$$\Delta\phi_{ql}^2 = \frac{1}{2\sqrt{N_p}}$$



$$\Delta\phi_{ql}^2 = \frac{1}{2\sqrt{N_p/2}}$$

H. M . Wiseman, 2002

Minimum phase fluctuation due to amplifier noise



Output of the amplifier:

$$E_{out}(t) = \sqrt{GP_{in}}\sin[\omega_0 t + \phi(t)] + n_a(t)$$

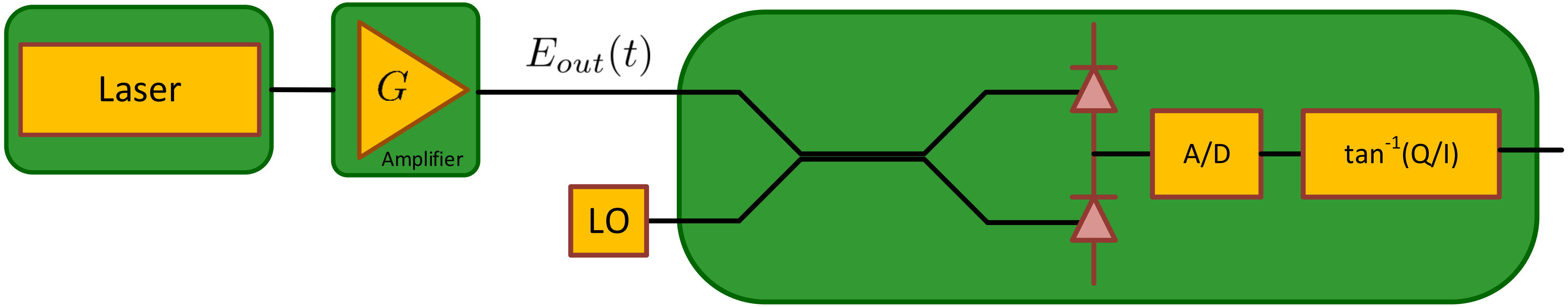
$n_a(t)$: Gaussian noise term added by the amplifier with variance :

$$\sigma_N^2 = P_N = h\nu(G-1)B$$

Quantum limited (minimum) phase fluctuation due to amplifier noise $(B = \Delta\nu_{FWHM})$:

$$\boxed{\Delta\phi_a^{MAP} = \frac{1}{2\sqrt{\sqrt{N_p/2}}} = \frac{1}{\sqrt{2\sqrt{GP_{in}/2h\nu(G-1)\pi\Delta\nu_{FWHM}}}}}$$

Optimum phase measurement for *high* SNR



Discrete-time signal after analogue-to-digital converter (A/D):

$$y[k] = 2R\sqrt{GP_{in}P_{LO}}\cos(\Delta\omega kT_s + \phi[k]) + n^{sh}[k] + n^{b}[k]$$

Phase estimation method: $\phi_{\tan^{-1}} = \arg[(y[k] + j\mathcal{H}\{y[k]\})e^{-j\Delta\omega t}] = \arg[(I + jQ)e^{-j\omega t}] = \tan^{-1}(Q/I)$

Quantum limited (minimum) phase fluctuation due to amplifier noise:

$$\boxed{\Delta\phi_a = \frac{1}{2\sqrt{N_p/2}} = \frac{1}{2\sqrt{GP_{in}/2h\nu(G-1)B}} = \frac{1}{2\sqrt{SNR/2}}}$$

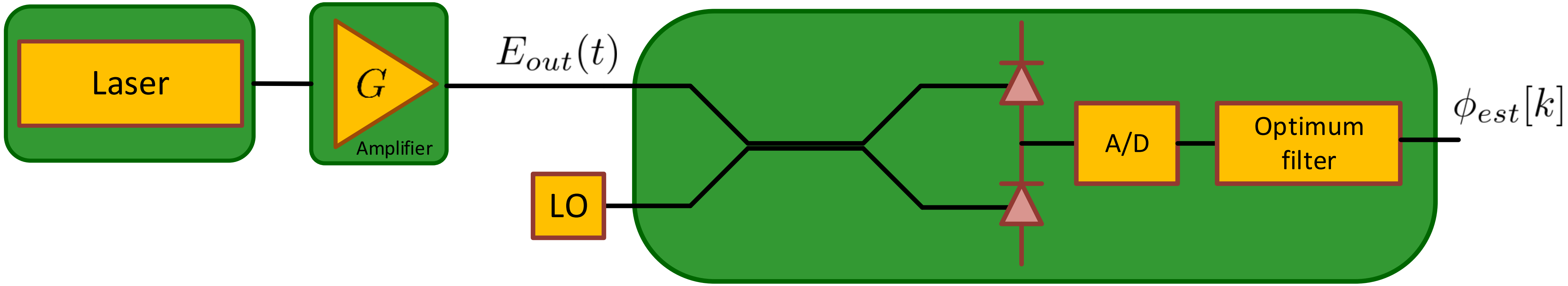
Measuring at low and medium SNR is important

- Technical noise dominates laser phase noise at low frequencies
- Determining fundamental laser linewidth (quantum noise limited) requires measuring beyond MHz
- Laser power may be low (output of the cavity)
- Frequency comb lines may have low power
- Several stages of amplification may reduce SNR

Signal-to-noise ratio of beat signal after heterodyne detection:

$$\boxed{SNR = \frac{2RP_sP_{LO}}{\sigma^2_{shot}+\sigma^2_b} = \frac{2RP_sP_{LO}}{2qRP_{LO}B+4P_{LO}N_Ah\nu(G-1)B}}$$

Optimum filtering for *wide-range* of SNRs



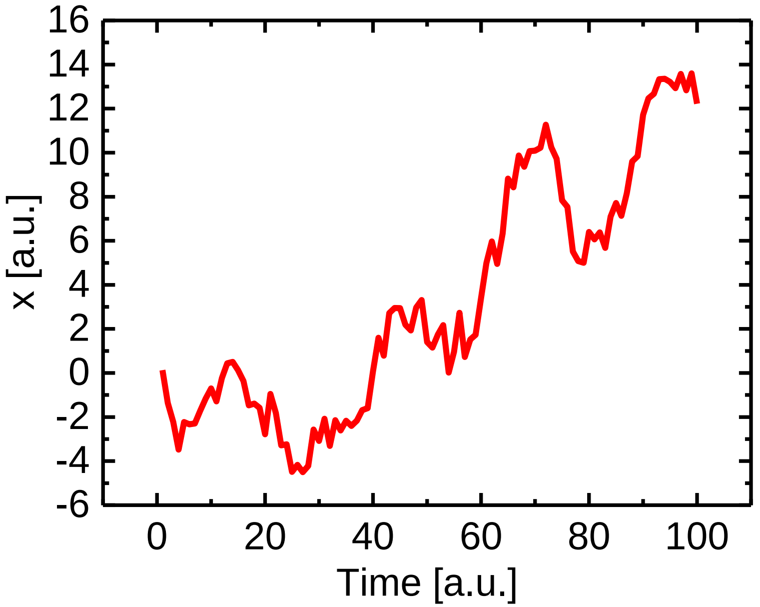
Given discrete-time signal after analogue-to-digital converter (A/D):

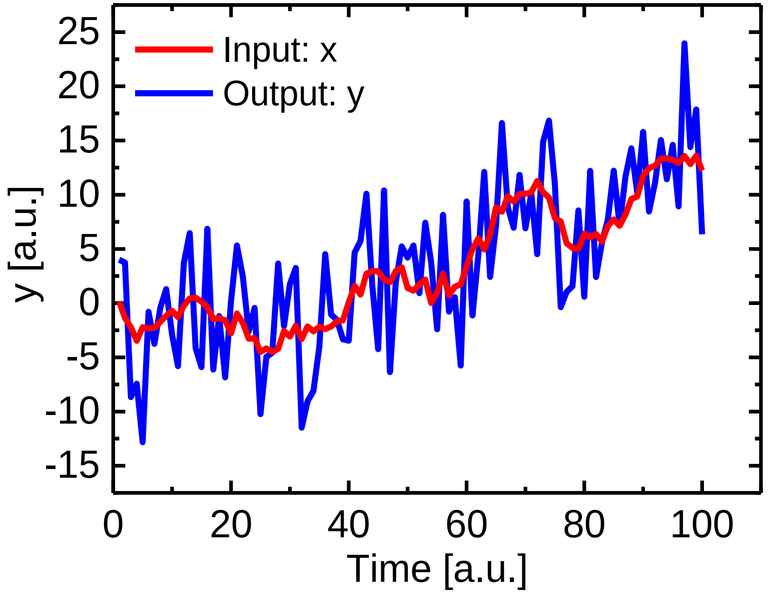
$$y[k] = 2R\sqrt{GP_{in}P_{LO}}\cos(\Delta\omega kT_s + \phi[k]) + n^{sh}[k] + n^{b}[k]$$

Optimum filter finds phase that is closest to $\phi[k]$ for a given SNR

For extracting signal from noise Bayesian filter is theoretically optimum filter
Kalman filter is an approximation of Bayesian filter

Bayesian filtering: infer *input x* from *noisy output y*





$$x_t = g(x_{t-1}, w_{t-1}) \rightarrow f(\cdot,\cdot) \rightarrow y_t = f(x_t, n_t)$$

States: amp., phase noise, PMD x_t

Mapping function: $g(\cdot)$

Process noise: w_{t-1}

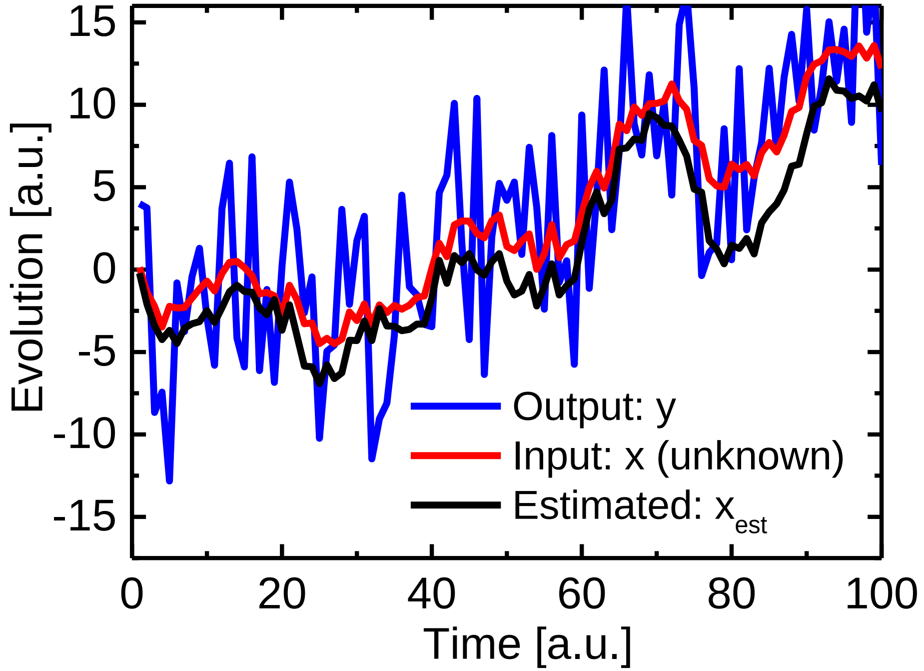
Observations: y_t

Mapping function: $f(\cdot)$

Measurement noise: n_t

S. Sarka, 2013

Bayesian filtering: infer *input x* from *noisy output y*



Given a measurement:

$$y = x + n$$

Compute:

$$\overbrace{p(x|y)}^{posterior} = \frac{p(y|x)\overbrace{p(x)}^{prior}}{p(y)} \longrightarrow x_{est} = E[x] = \int xp(x|y)dx$$

Bayesian filtering equations

Deterministic state space model:

$$x_t = g(x_{t-1}, w_{t-1})$$
$$y_t = f(x_t, n_t)$$

Probabilistic state space model:

$$\theta \sim p(\theta)$$
$$x_t \sim p(x_t | x_{t-1}, \theta)$$
$$y_t \sim p(y_t | x_t, \theta)$$

Use Kalman or particle filtering to solve:

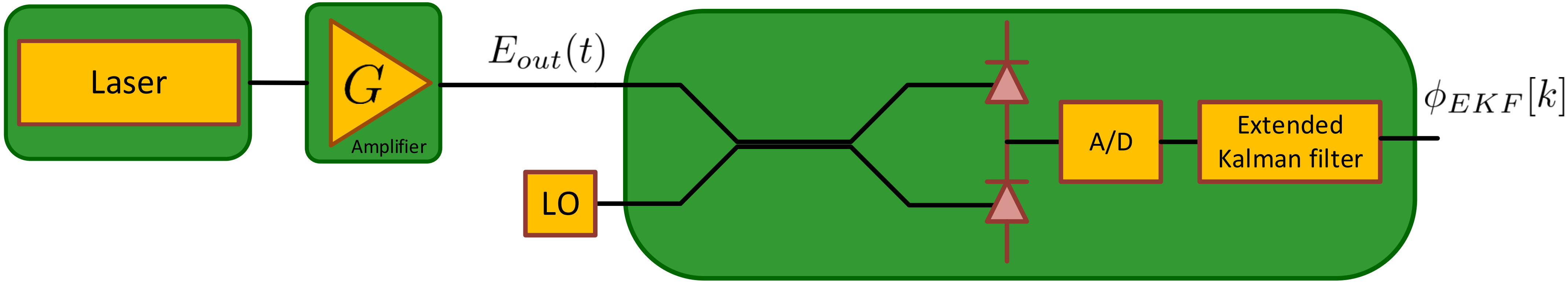
for t=1:T

1. Compute prior: $p(x_t | y_{1:t-1}) = \int p(x_t | x_t - 1) p(x_{t-1} | y_{1:t-1}) dx_{t-1}$

2. Compute posterior: $p(x_t | y_{1:t}) = \dfrac{p(y_t | x_t, \theta) p(x_t | y_{1:t-1}, \theta)}{p(y_{1:k} | \theta)}$

end

Approaching optimum filtering with Kalman filter



EKF based phase estimation approaches quantum limit:

$$\boxed{\sigma_{EKF} = \sqrt{\frac{1}{K}\sum_{k=1}^{K}(\phi^{true}[k] - \phi_{EKF}[k])^2} \rightarrow \Delta\phi^{MAP} = \frac{1}{\sqrt{2\sqrt{N_p/2}}}}$$

Practical implication of random phase fluctuations

Given a observation time *T*, laser phase noise variance is expressed as:

$$\boxed{\Delta\phi^2(T) = 2\pi\Delta\nu T}$$

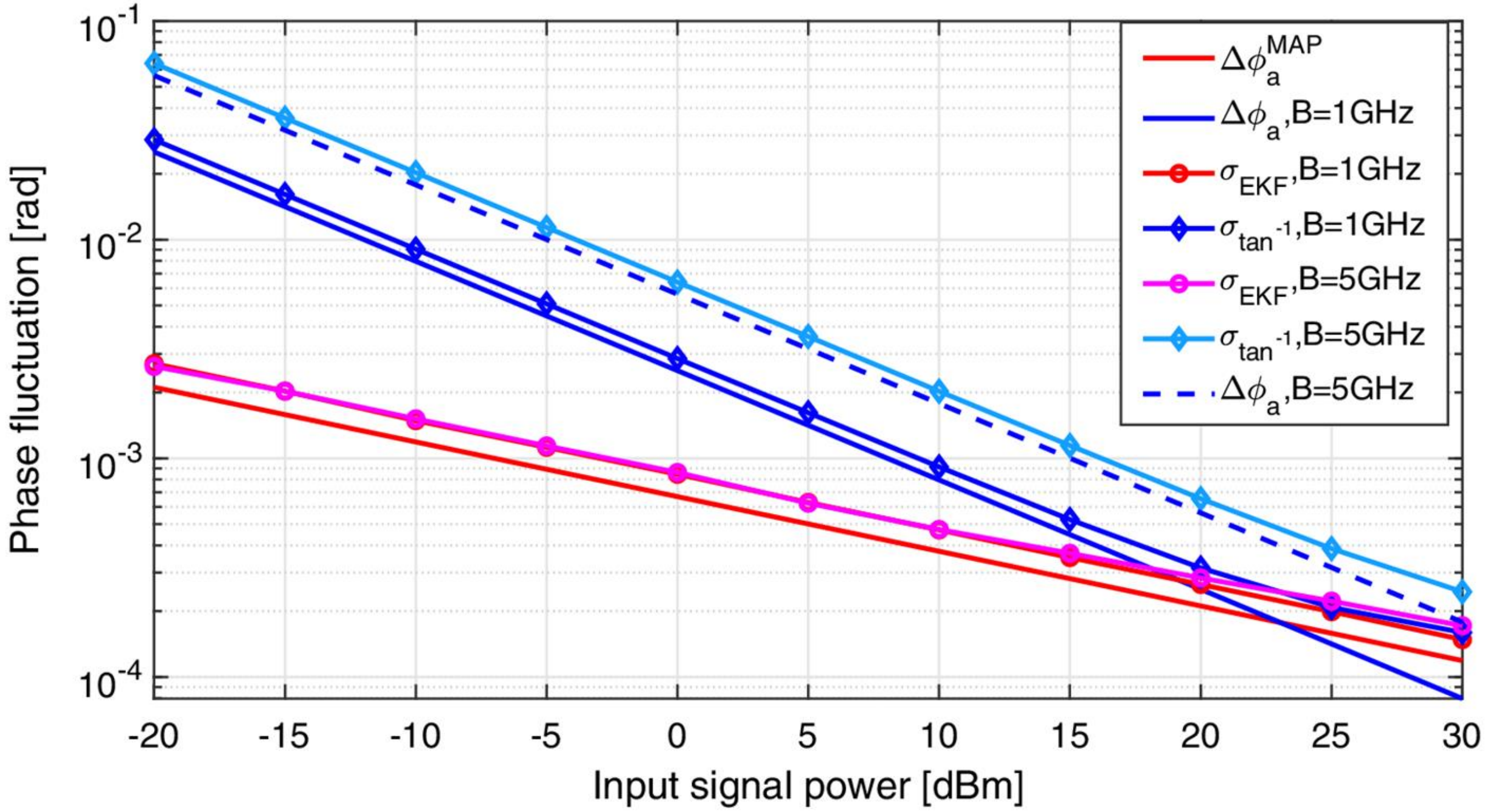
The corresponding *spectral broadening* expressed as:

$$\boxed{\Delta\nu = \frac{\Delta\phi^2(T)}{2\pi T}}$$

The quantum limited spectral broadening due to amplifier noise :

$$\boxed{\Delta\nu_a^{MAP} = \frac{\Delta\phi_a^{MAP}}{2\pi T}}$$

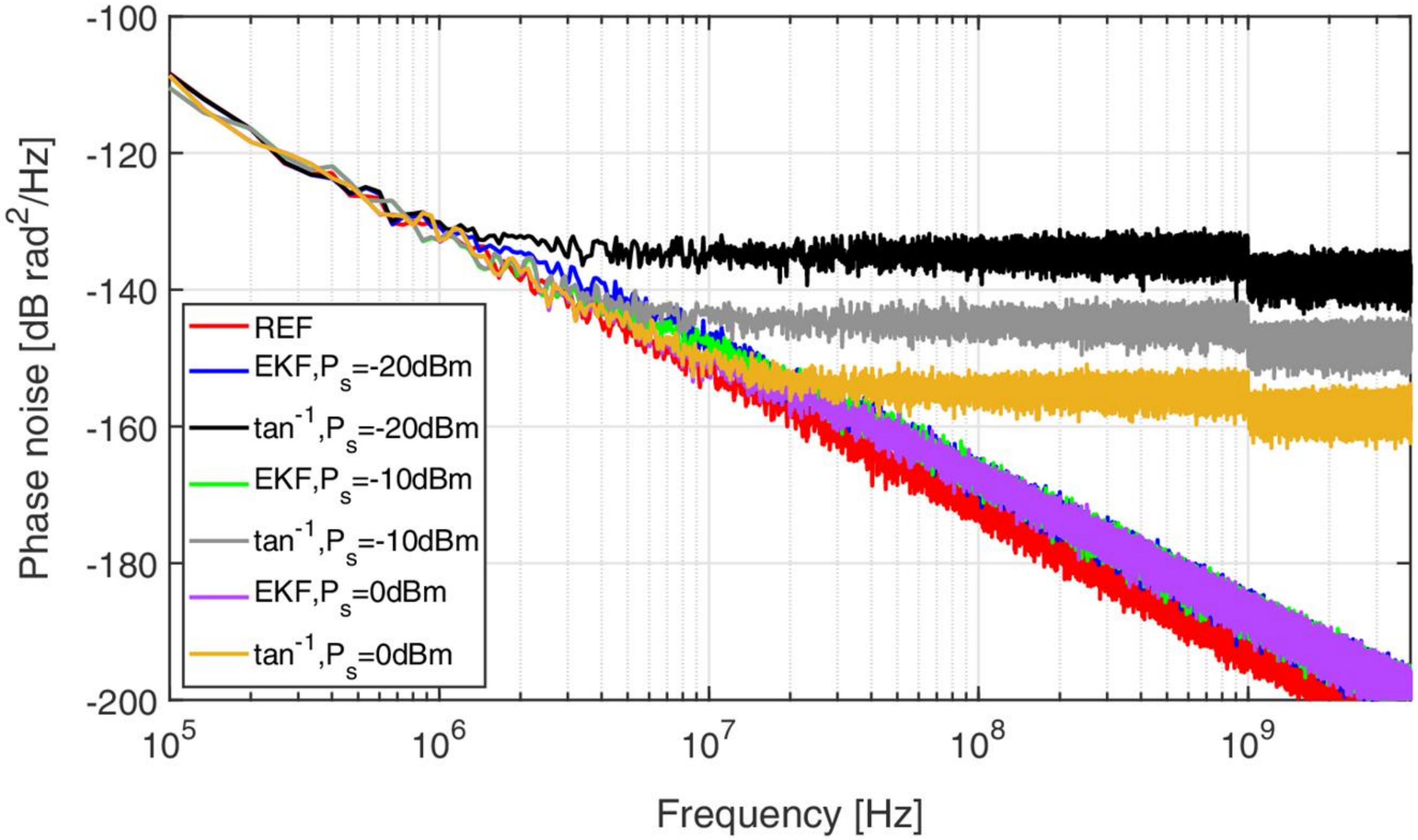
Numerical results: phase fluctuation



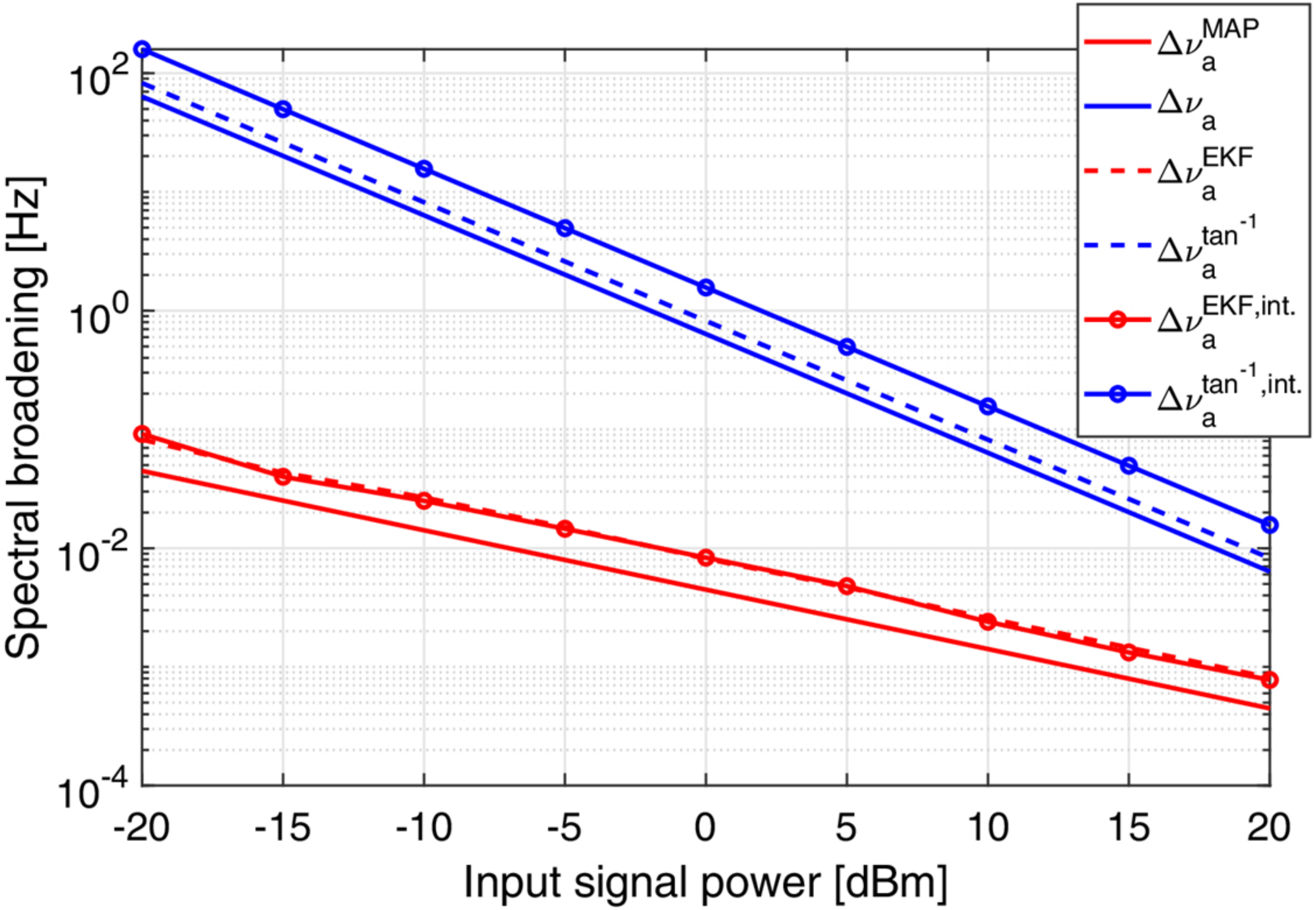
$$\Delta\nu_{FWHM} = 1 \text{ kHz}$$

$$\Delta\phi_a^{MAP} = \frac{1}{2\sqrt{\sqrt{N_p/2}}} = \frac{1}{\sqrt{2\sqrt{GP_{in}/2h\nu(G-1)\pi\Delta\nu_{FWHM}}}}$$

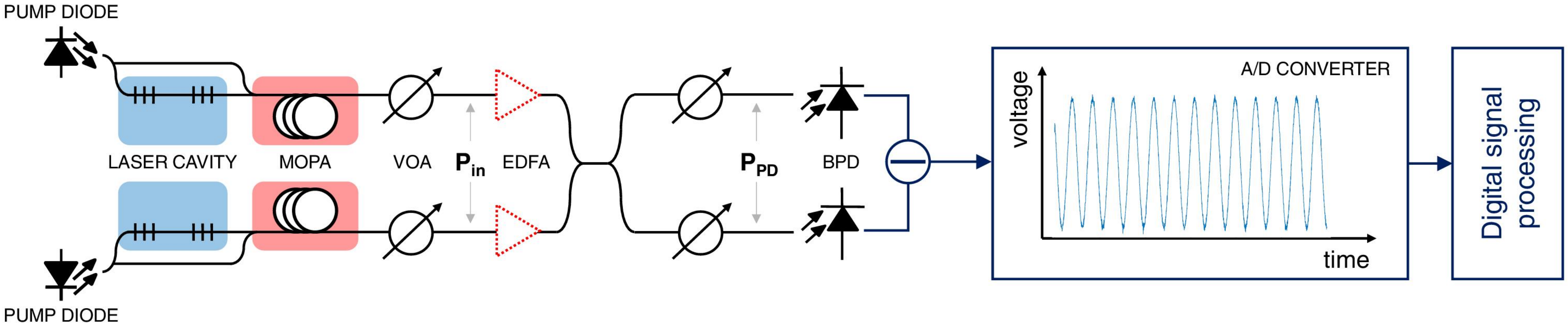
Numerical results: phase PSD



Numerical results: spectral broadening



Experimental set-up



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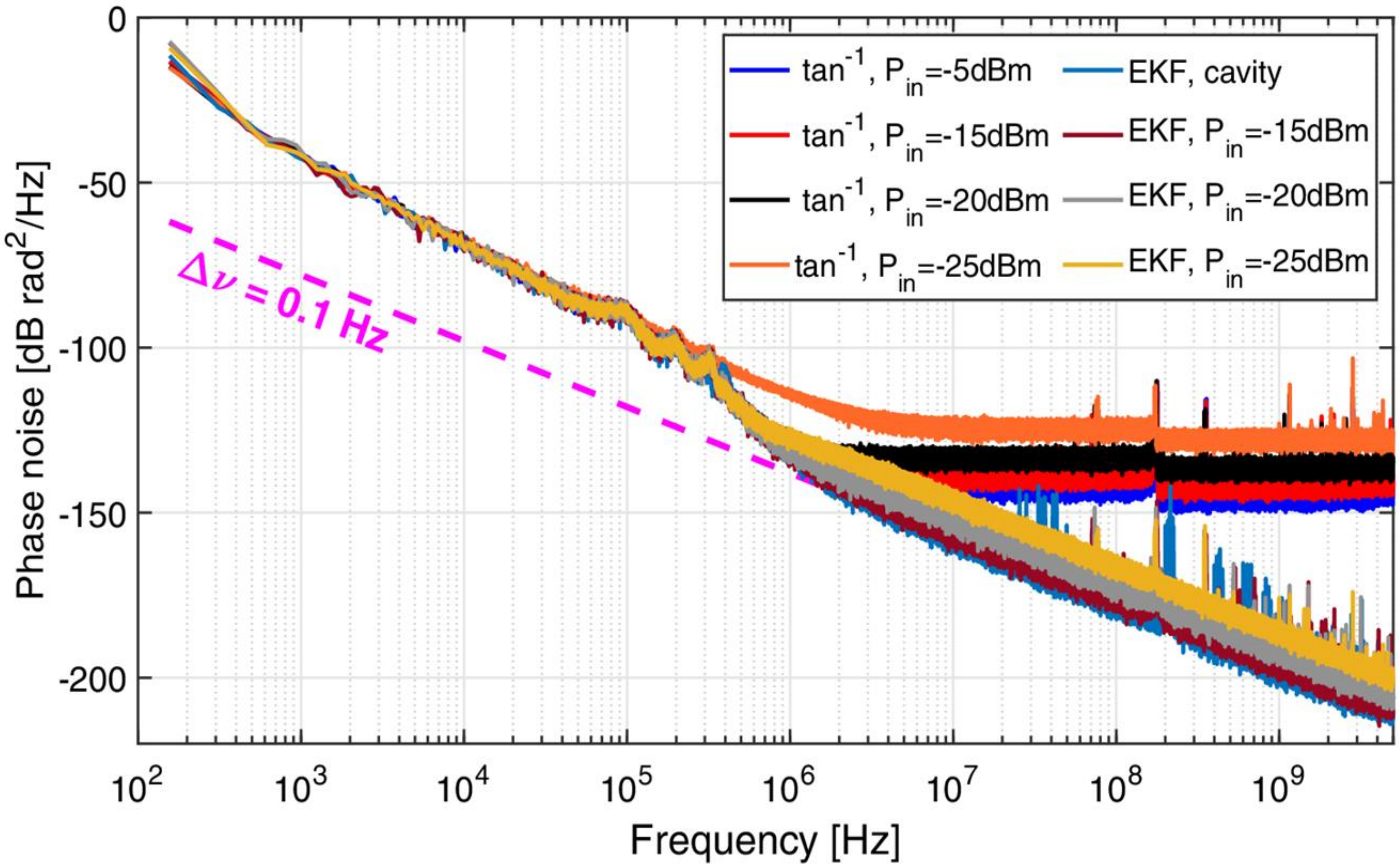
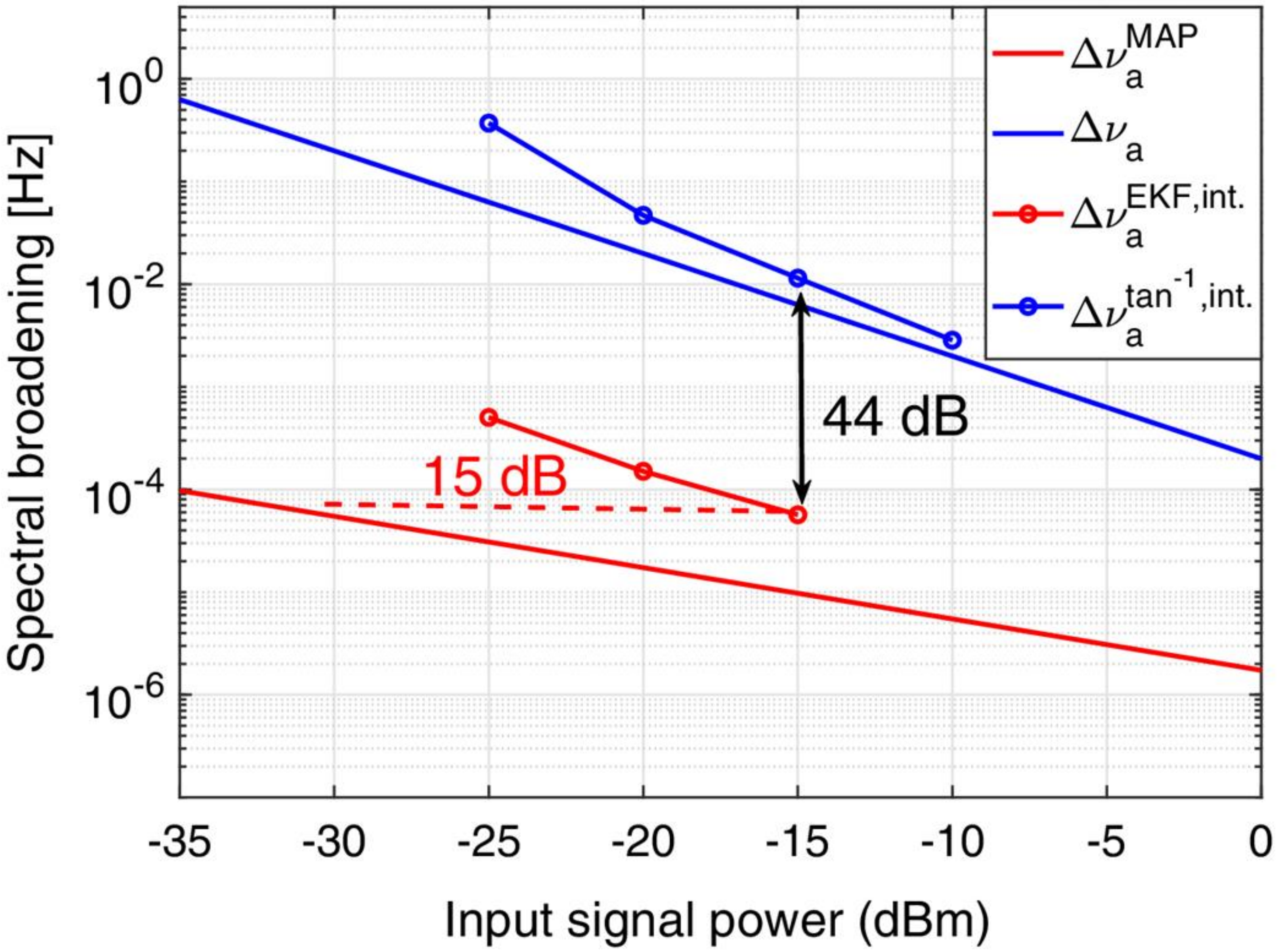
https://www.osapublishing.org/optica/abstract.cfm?uri=optica-8-10-1262

Experimental: phase-noise measurement

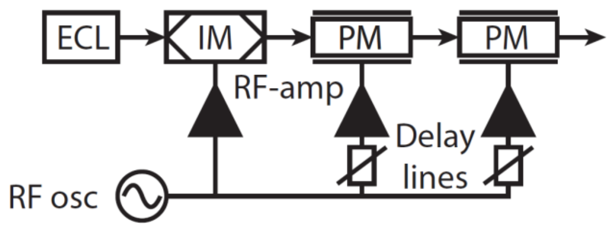


Experimental: spectral broadening



Optical frequency comb characterization

Electro optic comb generation



$$\phi_k^m = \phi_k^L + m\phi_k^{RF}$$

The phase noise of each line has two independent contributions

$$m = -\mathrm{H}, -\mathrm{H}+1, \dots, 0, \dots, \mathrm{H}-1, \mathrm{H}$$

Relative line index

$$\mathrm{H} = \left\lfloor \frac{M}{2} \right\rfloor + 1$$

Covariance of the phases

$$\mathrm{Var}[\phi_k^m] = \mathrm{Var}\left[\phi_k^L\right] + m^2 \mathrm{Var}\left[\phi_k^{RF}\right]$$

$$\mathrm{Cov}[\phi_k^m \phi_k^n] = \mathrm{Var}\left[\phi_k^L\right] + mn\mathrm{Var}\left[\phi_k^{RF}\right]$$

The covariance matrix describe how the noise variance affect different comb lines

$$\mathrm{Cov}\left[\boldsymbol{\phi}_k \boldsymbol{\phi}_k^\top\right] = \boldsymbol{\Sigma} = \boldsymbol{c}\boldsymbol{c}^\top \sigma_L^2 + \boldsymbol{h}\boldsymbol{h}^\top \sigma_{RF}^2$$

$$\boldsymbol{c} = \Big[\underbrace{1,1,\dots 1}_{M\ lines}\Big]^\top$$

$$\boldsymbol{h} = \Big[\underbrace{-\mathrm{H}, -\mathrm{H}+1, \dots, 0, \dots, \mathrm{H}-1, \mathrm{H}}_{M\ lines}\Big]^\top$$

Correlation of the phases

$$\mathrm{Corr}[\phi_k^m \phi_k^n] = \frac{\mathrm{Cov}[\phi_k^m \phi_k^n]}{\mathrm{std}[\phi_k^m]\mathrm{std}[\phi_k^m]}$$

The correlation matrix is re-scaled such that it’s maximum value is 1.

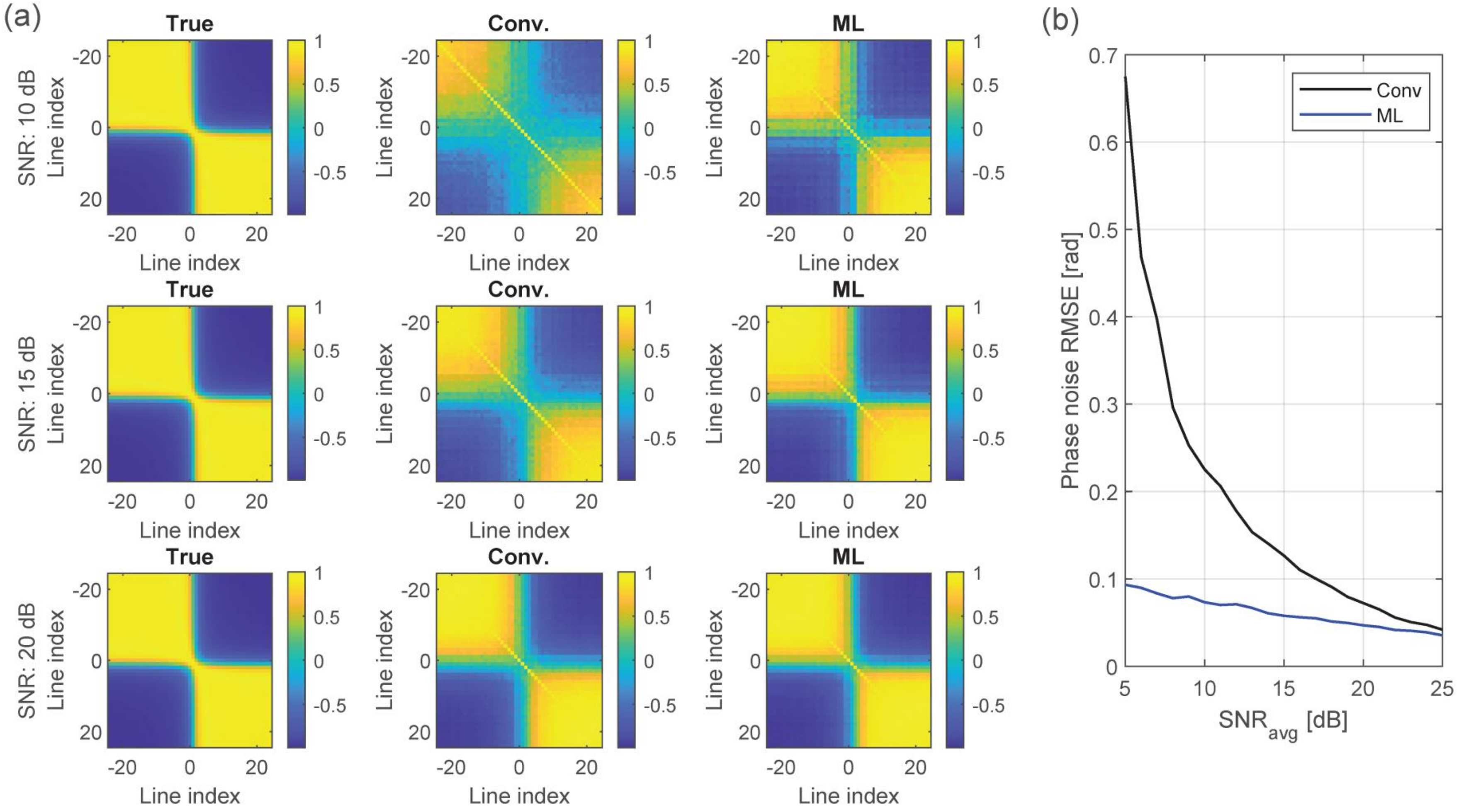
Correlation describes how similar are two lines.

1 = perfectly correlated lines

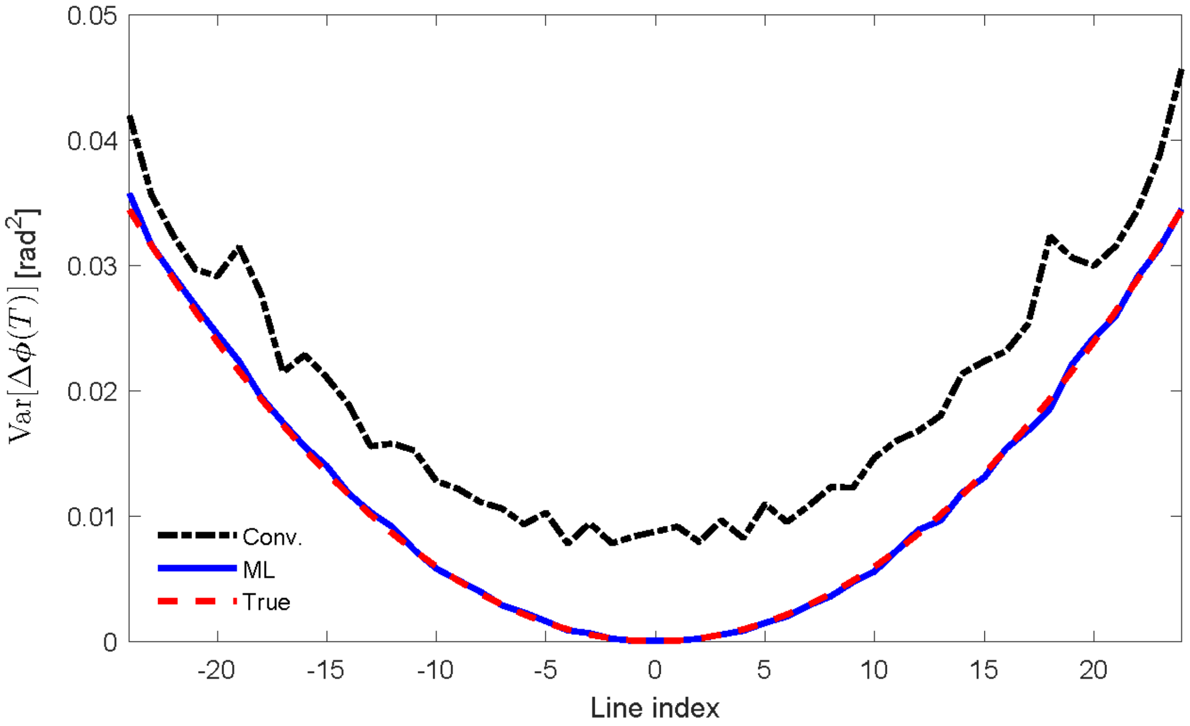
0 = uncorrelated lines

-1 = anticorrelated lines

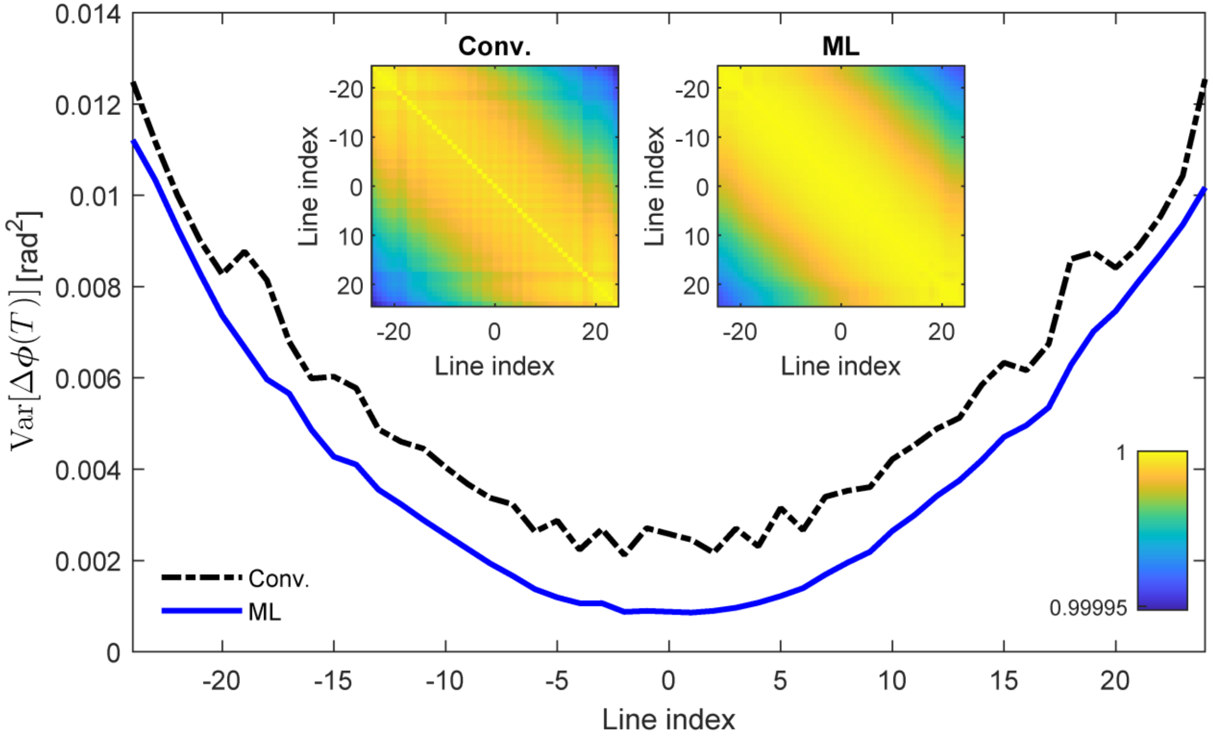
Frequency comb phase noise correlation matrix



Combs lines phase variance



(a) Simulations



(b) Experimental

Machine learning methods provides more accurate estimations

Conclusion and outlook

- Impact of amplifier noise on signal phase quantified
- Minimum impact of amplifier noise requires optimum (phase) detector
- Extended Kalman filtering approaches optimum detector
- Importance of optimum detector for measuring laser phase-noise demonstrated
- Measurement noise floor originates form non-optimal detector
- With optimum detector noise floor avoided $\Rightarrow$ fundamental laser linewidth revealed